Connectedness of the Continuum in Intuitionistic Mathematics

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Abstract: Working in INT (Intuitionistic analysis) we prove a strong, constructive connectedness property of the continuum: for any non-empty sets, A and B, if $\mathbb{R} = A \cup B$ then $A \cap B$ is non-empty. It is well known that the intuitionistic continuum is *indecomposable*: if $\mathbb{R} = A \cup B$ and $A \cap B = \emptyset$ then $A = \mathbb{R}$ or $A = \emptyset$, but this property is essentially negative—equivalent to if A, B are non-empty and $\mathbb{R} = A \cup B$ then $\neg(A \cap B \neq \emptyset)$. Our connectedness property is positive; so, given $a \in A$, $b \in B$, and a witness to $\mathbb{R} = A \cup B$, to prove our theorem we must construct a real number $r \in A \cap B$. We can construct the needed real number using only Bishop's constructive mathematics (BISH) and a weak form of Brouwer's continuity principle (and the choice principles that come from the BHK interpretation of quantifiers in constructive type theory). We also replace indecomposability by connectedness in some results of van Dalen that use additional intuitionistic axioms.

1 Introduction

In classical mathematics the continuum is *connected*: for non-empty *open* sets, A and B, if $\mathbb{R} = A \cup B$ then $A \cap B$ is non-empty. In intuitionistic mathematics we get a stronger result:

Theorem 1. *For* any *non-empty sets, A and B, if $\mathbb{R} = A \cup B$ then $A \cap B$ is non-empty.*

Of course, in constructive mathematics, the assumption $\mathbb{R} = A \cup B$ is stronger than it is in classical mathematics because the constructive meaning of $\forall r:\mathbb{R}.\ A(r) \vee B(r)$ is a function d of type $r:\mathbb{R} \to A(r)+B(r)$ that decides whether r is in A or in B (and provides the witness). However, it is not obvious that we can use the function d and real numbers $a \in A$ and $b \in B$ to construct a real number $r \in A \cap B$. Using a weak form of Brouwer's continuity principle we can, indeed, construct such a real number.

Nuprl implements constructive type theory with Bar Induction and a Continuity principle. This makes Nuprl suitable for constructing formal proofs in INT (Intuitionistic mathematics). Presenting a detailed proof of Theorem 1 gives us an opportunity to review how the basic definitions in Bishop's constructive analysis are formalized in Nuprl, and how the continuity principle is stated (and why it is true in Nuprl). The other main feature of INT, Bar Induction, will not be needed to prove our theorem.

2 The Real Numbers

Bishop defined a real number to be sequence of rational numbers $q_1, q_2, q_3 \ldots$ such that

$$\forall n, m\colon \mathbb{N}^+.\ |q_n - q_m| \leq n^{-1} + m^{-1} \qquad (\mathbb{N}^+ = \{1, 2, 3, \ldots\})$$

We use a variant of Bishop's definition that results from normalizing the rationals q_n to always have denominator $2n$ and then clearing the denominators in Bishop's regularity condition. This allows us to define the reals directly from the integers and to define all operations on reals using computations on integers. Of course, computations on rationals also reduce to computations on integers, but, after first implementing Bishop's original definitions (which were not meant to be efficient) we discovered that because they used arithmetic on rationals, they were unnecessarily exact. In our variant definitions each operation on reals rounds off the n^{th} approximation so that it corresponds to a rational approximation with denominator of $2n$. This rounding avoids unnecessary accuracy and turns out to make the algorithms more efficient.

For the connectedness property (Theorem 1) we prove in this paper, efficiency of computation is not very important. However, the definition of real numbers as regular sequences of integers makes the use of the continuity principle more straightforward. Also, some of the techniques we developed for constructing reals defined as integer sequences (viz. *acceleration*) helped us in the proof of Theorem 1. Because there is a straightforward translation between Bishop's real numbers and the reals we use, our proof of Theorem 1 could be easily converted to use Bishop's definitions.

Definition 2. A sequence $x_1, x_2, x_3, \dots$ of integers is regular if

$$\forall n, m\!:\!\mathbb{N}^+.\ |m * x_n - n * x_m| \le 2(n+m)$$

The sequence is k-regular if

$$\forall n, m\!:\!\mathbb{N}^+.\ |m * x_n - n * x_m| \le 2k(n+m)$$

Definition 3. $\mathbb{R} = \{x : \mathbb{N}^+ \to \mathbb{Z} \mid x \text{ is regular}\}$

To better understand what the regularity condition means we prove the following easy lemma:

Lemma 4. *Sequence x is k-regular if and only if for every $n, m \in \mathbb{N}^+$, the rational intervals $[\frac{x_n}{2kn} - \frac{1}{n}, \frac{x_n}{2kn} + \frac{1}{n}]$ and $[\frac{x_m}{2km} - \frac{1}{m}, \frac{x_m}{2km} + \frac{1}{m}]$ overlap (i.e. have a common member $q \in \mathbb{Q}$).*

Proof. The intervals overlap if and only if $\frac{x_n}{2kn} + \frac{1}{n} \ge \frac{x_m}{2km} - \frac{1}{m}$ and $\frac{x_n}{2kn} - \frac{1}{n} \le \frac{x_m}{2km} + \frac{1}{m}$. Multiplying by $2knm$ to clear the denominators, these hold if and only if $m*x_n + 2km \ge n*x_m - 2kn$ and $m*x_n - 2km \le n*x_m + 2kn$, which is the same as $2kn + 2km \ge n*x_m - m*x_n$ and $m*x_n - n*x_m \le 2km + 2kn$. These hold if and only if $|m * x_n - n * x_m| \le 2k(n+m)$. □

Remark The k-regularity condition does not say that the intervals $\frac{x_n}{2kn} \pm \frac{1}{n}$ are nested. It only says that they are pairwise overlapping. But if x is k-regular (even for just $n, m < B$) we can find a rational q that is a member of all the intervals $\frac{x_n}{2kn} \pm \frac{1}{n}$ for $n < B$, for we can take $q = \min\{\frac{x_n}{2kn} + \frac{1}{n} \mid n < B\}$.

2.1 Equivalence

Definition 5. Real numbers x and y are equivalent, $(x \equiv y)$, if and only if $\forall n\!:\!\mathbb{N}^+.\ |x_n - y_n| \le 4$

We show next that $(x \equiv y)$ is, in fact, an equivalence relation on $\mathbb{R}$. The proof is similar to Bishop's proof but introduces some concepts that we will need later. Following Bishop, we usually write the equivalence as $x = y$, and Nuprl displays the relation that way, but in this paper we need to distinguish between equality of regular sequences and the equivalence relation on regular sequences. So we will use $x = y$ for the former and $x \equiv y$ for the latter.

Definition 6. The relation $\text{bnddiff}(x, y) \equiv \exists B : \mathbb{N}.\ \forall n : \mathbb{N}^+.\ |x_n - y_n| \le B$ says that the pointwise difference between x and y is bounded.

Lemma 7. *For all $x, y \in \mathbb{R}$, $(x \equiv y) \Longleftrightarrow \text{bnddiff}(x, y)$*

Proof. If $(x \equiv y)$ then we may take $B = 4$. In the other direction, suppose $\forall n : \mathbb{N}^+.\ |x_n - y_n| \leq B$, then, for any $n, m \in \mathbb{N}^+$, if $5 \leq |x_n - y_n|$ then $5m \leq m|x_n - y_n| \leq |m * x_n - n * x_m| + |n * x_m - n * y_m| + |n * y_m - m * y_n|$, by the triangle inequality. By regularity and our assumption, the right side is $\leq 4(n + m) + nB$. Thus, $5m \leq 4(n + m) + nB$, so, $m \leq (4 + B)n$ and we obtain a contradiction when $m = 1 + (4 + B)n$. Hence, $|x_n - y_n| \leq 4$. Note that this proof by contradiction is constructive because, since $|x_n - y_n| \in \mathbb{Z}$, we can prove $(5 \leq |x_n - y_n|) \vee (|x_n - y_n| \leq 4)$. $\square$

Corollary 8. $\equiv$ *is an equivalence relation on* $\mathbb{R}$

Proof. The bounded-difference relation, $\text{bnddiff}(x, y)$ is clearly an equivalence relation on integer sequences. $\square$

2.2 Acceleration

We use the symbol $\div$ for integer division; it satisfies:

$$n = k(n \div k) + (n \text{ rem } k) \quad (\text{with } |n \text{ rem } k| < |k|)$$

Definition 9. The sequence $\text{accel}(k, x) = \lambda n.\ x_{2kn} \div 2k$ is called the k-acceleration of sequence x.

Lemma 10. *If sequence x is k-regular, then $accel(k, x)$ is regular, and $bnddiff(accel(k, x), x)$.*

Proof. Because x is k-regular, $2k|m * x_{2kn} - n * x_{2km}| \leq 2k(2kn + 2km)$, so $|m * x_{2kn} - n * x_{2km}| \leq (2kn + 2km)$. Thus,

$$\begin{aligned}
& 2k|m * (x_{2kn} \div 2k) - n * (x_{2km} \div 2k)| \\
= \ & |m * 2k * (x_{2kn} \div 2k) - n * 2k * (x_{2km} \div 2k)| \\
\leq \ & |m * x_{2kn} - n * x_{2km}| + m|x_{2kn} \text{ rem } 2k| + n|x_{2km} \text{ rem } 2k| \\
\leq \ & (2kn + 2km) + 2k(n + m) \\
= \ & 4k(n + m)
\end{aligned}$$

and hence, $|m * \text{accel}(k, x)_n - n * \text{accel}(k, x)_m| \leq 2(n + m)$, so $\text{accel}(k, x)$ is regular.

To show that the difference between $\mathrm{accel}(k, x)$ and x is bounded,

$$\begin{aligned} & 2kn|\mathrm{accel}(k, x)_n - x_n| \\ = \; & |n * 2k * x_{2kn} \div 2k - 2kn * x_n| \\ \leq \; & |n * x_{2kn} - 2kn * x_n| + n|x_{2kn} \text{ rem } 2k| \\ \leq \; & 2k(n + 2kn) + 2kn \\ = \; & 2kn(2 + 2k) \end{aligned}$$

hence, $|\mathrm{accel}(k, x)_n - x_n| \leq (2 + 2k)$, so we can take $B = (2 + 2k)$ in the definition of $\mathrm{bnddiff}(\mathrm{accel}(k, x), x)$. □

Corollary 11. *If $x \in \mathbb{R}$ and $k \in \mathbb{N}^+$ then $\mathit{accel}(k, x) \equiv x$*

Proof. x is regular and therefore k-regular, so $y = \mathrm{accel}(k, x)$ is regular and $\mathrm{bnddiff}(y, x)$. So, by Lemma 7, $y \equiv x$. □

2.3 Arithmetic and Ordering

If x and y are regular sequences then it is easy to see that the pointwise sum $\lambda n.\ x_n + y_n$ is a 2-regular sequence. So we define addition on real numbers to be

$$x + y = \mathrm{accel}(2, \lambda n.\ x_n + y_n)$$

Absolute value and unary minus do not need acceleration, the special case of division by a positive natural number $k \in \mathbb{N}^+$ has a simple definition, and the orderings, $x \leq y$ and $x < y$, are defined as follows:

$$|x| = \lambda n.\ |x_n| \tag{1}$$

$$-x = \lambda n. -x_n \tag{2}$$

$$x/k = \lambda n.\ x_{2n} \div 2k \tag{3}$$

$$x \leq y \Leftrightarrow \forall n{:}\mathbb{N}^+.\ x_n \leq y_n + 4 \tag{4}$$

$$x < y \Leftrightarrow \exists n{:}\mathbb{N}^+.\ x_n + 4 < y_n \tag{5}$$

For every integer z we get a real $\lambda n.\ 2zn$ that we will also write as z. It follows from these definitions that

$$|x - (x_m/2m)| \leq 1/m \tag{6}$$

All of these operations and relations must be shown to respect the equivalence relation. For example,

$$\begin{aligned} x_1 \equiv x_2 \wedge y_1 \equiv y_2 & \Rightarrow (x_1 + y_1) \equiv (x_2 + y_2) \\ x_1 \equiv x_2 \wedge y_1 \equiv y_2 & \Rightarrow (x_1 \leq y_1) \Leftrightarrow (x_2 \leq y_2) \end{aligned}$$

These proofs are straightforward using the definitions and Lemma 7. They have all been done in Nuprl along with much more. Everything in chapter 2 of Bishop and Bridges "Constructive Analysis"[1], has been formalized; essentially the material of a good one semester calculus course. One particular theorem we will need is that a sequence of real numbers converges if and only if it is a Cauchy sequence.

3 Beginning the proof of Theorem 1

We can now start the proof of the main theorem, but first we need a formal statement of it.

In constructive type theory, propositions are identified with types and a witness to the truth of a proposition is just a member of that type. To avoid Russel's paradox, the types are members of a cumulative hierarchy of *universes* $\mathbb{U}_i$ where for $i < j$ we have $\mathbb{U}_i \subset \mathbb{U}_j$ and $\mathbb{U}_i \in \mathbb{U}_{i+1}$. When the universe level i is irrelevant, then we write $Type$ instead of $\mathbb{U}_i$, and when we are thinking of the types as propositions then we write $\mathbb{P}_i$ or just $\mathbb{P}$.

A proposition P about real numbers is then a member of the type $\mathbb{R} \to \mathbb{P}$. Then for any $x \in \mathbb{R}$, $P(x)$ is a proposition. But note that $P(x)$ is a proposition about x as a regular sequence of integers. It could be something as simple as $x(1) < 10$. A proposition like that would not respect the equivalence relation $x \equiv y$ because there are regular sequences $x \equiv y$ where $x(1) = 9$ and $y(1) = 10$.

Bishop reserves the word *set* (of real numbers) to mean a proposition that respects the equivalence relation.

Definition 12. $P \in \mathbb{R} \to \mathbb{P}$ is a *set* of reals (and we write $\mathrm{Set}(P)$) if
$\forall x\!:\!\mathbb{R}.\ \forall y\!:\!\{\mathbb{R}|\ x \equiv y\}.\ P(x) \Rightarrow P(y)$

Similarly, f of type $\mathbb{R} \to T$, where T is a type, is called an *operation* on $\mathbb{R}$. Bishop reserves the word *function* for operations that respect the equivalence relation. The precise meaning of this depends on the type T. If T is a "discrete" type like $\mathbb{B}$ or $\mathbb{Z}$ then it means that $x \equiv y \Rightarrow f(x) = f(y)$. But if T is $\mathbb{R}$ then it means $x \equiv y \Rightarrow f(x) \equiv f(y)$.

The formal statement of our main theorem is therefore:

Theorem 13. *(Main theorem restated) For all $A, B : \mathbb{R} \to \mathbb{P}$, if $Set(A)$ and $Set(B)$ and $\exists a\!:\!\mathbb{R}.\ A(a)$ and $\exists b\!:\!\mathbb{R}.\ B(b)$ then*

$$(\forall r\!:\!\mathbb{R}.\ A(r) \vee B(r)) \Rightarrow (\exists r\!:\!\mathbb{R}.\ A(r) \wedge B(r))$$

Proof. Let a_0 and b_0 be reals such that $A(a_0)$ and $B(b_0)$.

The assumption $\forall r\!:\!\mathbb{R}.\ A(r) \vee B(r)$ gives us a *operation* d of type

$$d \in r\!:\!\mathbb{R} \to A(r) + B(r)$$

The operation $bd = \lambda r.\ isl(d(r))$ then has type $\mathbb{R} \to \mathbb{B}$ and

$$\begin{aligned}(bd(r) = \text{tt}) &\Rightarrow A(r)\\ (bd(r) = \text{ff}) &\Rightarrow B(r)\end{aligned}$$

Now if $bd(a_0) = \text{ff}$ then $B(a_0) \wedge A(a_0)$ and we are done, and similarly if $bd(b_0) = \text{tt}$. We need consider only the remaining case $bd(a_0) = \text{tt}$ and $bd(b_0) = \text{ff}$.

Note that the operation bd need not be a *function* in Bishop's sense. If it were, our job would be easy because we could invoke the corollary to Brouwer's uniform continuity theorem that all *functions* in $\mathbb{R} \to \mathbb{B}$ are constant to reach a contradiction in our remaining case. But we will have to work harder.

Now we define by primitive recursion a sequence of pairs $\langle a_n, b_n\rangle$ starting with our given $\langle a_0, b_0\rangle$. To define $\langle a_{n+1}, b_{n+1}\rangle$ we let $m_n = (a_n + b_n)/2$ and

$$\langle a_{n+1}, b_{n+1}\rangle = \text{if } bd(m_n) \text{ then} \langle m_n, b_n\rangle \text{ else } \langle a_n, m_n\rangle$$

By induction we see that for all $n \in \mathbb{N}$

$$bd(a_n) = \text{tt} \wedge bd(b_n) = \text{ff} \wedge |a_n - b_n| = |a_0 - b_0|/2^n$$

and also for all $m > n$

$$|a_n - a_m| \leq |a_n - b_n| \wedge |b_n - b_m| \leq |a_n - b_n|$$

Therefore the sequences a_n and b_n are Cauchy sequences and they converge to the same real number x.

We now state our main lemma and use that to finish the proof of the main theorem. The proof of the main lemma will require the continuity principle, so we will have to introduce that principle before we finish the proof of the main lemma.

Lemma 14. *(Main lemma) For all $x \in \mathbb{R}$ there exists $x' \in \mathbb{R}$ such that $x' \equiv x$ and for any sequence g_n converging to x and any boolean operation $P \in \mathbb{R} \to \mathbb{B}$, there exist $n \in \mathbb{N}$ and $z \in \mathbb{R}$ such that $z \equiv g_n$ and $P(z) = P(x')$.*

To finish the proof of the main theorem we use this lemma with real x the common limit of sequences a_n and b_n.

If $bd(x') = \text{tt}$ then we take $g_n = b_n$ and get $bd(z) = \text{tt}$ for some real $z \equiv b_n$, for some n. Therefore $A(z)$ is true and, because $\text{Set}(A)$, also $A(b_n)$. But $B(b_n)$ because $bd(b_n) = \text{ff}$, so b_n is in both sets A and B and we are done in this case.

If $bd(x') = \text{ff}$ then we take $g_n = a_n$ and get $bd(z) = \text{ff}$ for some real $z \equiv a_n$ for some n. Then $B(z)$ and hence $B(a_n)$, because $\text{Set}(B)$. But also, $A(a_n)$, because $bd(a_n) = \text{tt}$, so a_n is in both sets A and B.

□

4 The Continuity principle

Suppose that we have a functional F of type $(\mathbb{N}^+ \to \mathbb{Z}) \to \mathbb{B}$. Then F takes a sequence of integers as input and uses it to compute a boolean output. The continuity principle asserts that such a computation uses only a finite amount of the input sequence. Therefore, if for given $f \in (\mathbb{N}^+ \to \mathbb{Z})$ and $b \in \mathbb{B}$ we have $F(f) = b$, then there is a bound $k \in \mathbb{N}^+$ such that for any sequence $g \in \mathbb{N}^+ \to \mathbb{Z}$ if $g(n) = f(n)$ for all $1 \le n < k$ then $F(g) = b$.

We use the type $\mathbb{N}^+_k = \{n : \mathbb{N}^+ \mid n < k\}$ to help state the continuity principle succinctly. We also introduce the abbreviations $\mathbb{S} = \mathbb{N}^+ \to \mathbb{Z}$ and $\mathbb{S}_k = \mathbb{N}^+_k \to \mathbb{Z}$. Then the condition $g(n) = f(n)$ for all $1 \le n < k$ is the same as $g = f \in \mathbb{S}_k$. Using these abbreviations, one possible statement of the continuity principle is

$$\forall F{:}\mathbb{S} \to \mathbb{B}.\ \forall f{:}\mathbb{S}.\ \exists k{:}\mathbb{N}^+.\ \forall g{:}\mathbb{S}.\ (g = f \in \mathbb{S}_k) \Rightarrow F(g) = F(f)$$

Unfortunately, this version of the principle is provably false in constructive type theory. We need not go into the reason for this here; so we refer the interested reader to papers by Escardo and Xu [2] and Rahli and Bickford [6]. However a weaker version of this principle is derivable from a continuity rule that is true in Nuprl. That rule has been proven correct by Rahli (using the Coq theorem prover) from the formal definition of Nuprl.

Suppose that for a given sequence $f \in \mathbb{S}$ we have a family of sequences $g_k \in \mathbb{S}$ where, for every $k \in \mathbb{N}^+$, $g_k = f \in \mathbb{S}_k$. Then the continuity principle should imply that for at least one k we have $F(g_k) = F(f)$. This version of the continuity principle is all we will need to prove Theorem 1, and it (and many stronger principles) is derivable from the continuity rule proved correct by Rahli. For the purposes of this paper we can simply accept the following axiom for INT.

Axiom 15. *(Weak Continuity) For any* $F \in \mathbb{S} \to \mathbb{B}$ *and any* $f \in \mathbb{S}$,

$$\forall G\!:\!k\!:\!\mathbb{N}^+ \to \{g\!:\!\mathbb{S} \mid g = f \in \mathbb{S}_k\}.\ \exists k\!:\!\mathbb{N}^+.\ F(G(k)) = F(f)$$

Lemma 16. *(Regularization) There is a map reg* $\in \mathbb{S} \to \mathbb{R}$ *such that if* s *is regular (i.e.* $s \in \mathbb{R}$*), then reg*$(s) = s$.

Proof. Let s be any sequence in $\mathbb{S}$. For any $j \in \mathbb{N}^+$ we can decide whether the regularity condition $|ns_m - ns_n| \leq 2(n+m)$ holds for all $n, m \leq j$. If so, we let $\mathrm{reg}(s)_j = s_j$. If not, we let j_0 be the first failure of regularity and use the remark after the proof of Lemma 4 to get a rational $q = a/b$ that is a member of all the intervals $\frac{s_n}{2n} \pm \frac{1}{n}$ for $n < j_0$. Then, for $j \geq j_0$ we let $\mathrm{reg}_k(s)_j = q_j$. Then Lemma 4 shows that $r = \mathrm{reg}(s)$ is regular because q is in all the intervals $\frac{r_n}{2n} \pm \frac{1}{n}$. □

Using regularization, we derive from Axiom 15 the following:

Lemma 17. *(Weak Continuity for* $\mathbb{R}$*) For any* $x \in \mathbb{R}$ *and any* $P \in \mathbb{R} \to \mathbb{B}$

$$\forall G\!:\!k\!:\!\mathbb{N}^+ \to \{y\!:\!\mathbb{R} \mid y = x \in \mathbb{S}_k\}.\ \exists k\!:\!\mathbb{N}^+.\ P(G(k)) = P(x)$$

Proof. Use Axiom 15 with $\lambda s.\ F(\mathrm{reg}(s))$. □

5 Proof of Main Lemma

We are given $x \in \mathbb{R}$, $P \in \mathbb{R} \to \mathbb{B}$, and a sequence of reals g_n converging to x. To use Lemma 17 we need a family of reals $G(k)$ such that $G(k)$ agrees exactly with x up to k and is equivalent to some g_n. For sufficiently large n, g_n will agree closely with x up to k, but will not necessarily agree exactly (one need only think of $(2n-1)/2n$ converging to 1.)

To get a real number z that is equivalent to g_n and agrees exactly with x up to k we need to "blend" x and g_n.

Definition 18.

$$\mathrm{blend}(k, x, y) = \lambda n.\ \text{if } n < k \text{ then } x_n \text{ else } y_n$$

Lemma 19. *If* $k \in \mathbb{N}^+$, $x, y \in \mathbb{R}$ *and* $|x - y| \leq 1/6k$ *then blend*$(6k, x, y)$ *is a* 3*-regular sequence,* $z = \mathit{accel}(3, \mathit{blend}(6k, x, y))$ *is a real number,* $z \equiv y$*, and* $z = \mathit{accel}(3, x) \in \mathbb{S}_k$.

Proof. We have to prove

$$|m * \text{blend}(6k, x, y)_n - n * \text{blend}(6k, x, y)_m| \le 6(n + m)$$

This follows from regularity of x if $n < 6k$ and $m < 6k$ and from regularity of y if $6k \le n$ and $6k \le m$. So, by symmetry, it is enough to consider $n < 6k \le m$, in which case we must show $|m * x_n - n * y_m| \le 6(n + m)$.

$$\begin{aligned} & |x_m - y_m| \\ = \;& 2m|(x_m/2m) - (y_m/2m)| \\ \le \;& 2m(|(x_m/2m) - x| + |x - y| + |y - (y_m/2m)|) \\ \le \;& 2m(1/m + 1/6k + 1/m) \qquad \text{using inequality (6) and } |x - y| \le 1/6k \\ = \;& 4 + (2m/6k) \end{aligned}$$

So, $n|x_m - y_m| \le 4n + 2m(n/6k) \le 4n + 2m \le 4(n + m)$. Thus,

$$\begin{aligned} & |m * x_n - n * y_m| \\ \le \;& |m * x_n - n * x_m| + |n * x_m - n * y_m| \\ \le \;& 2(n + m) + n|x_m - y_m| \\ \le \;& 6(n + m) \end{aligned}$$

By Lemma 10, $z = \text{accel}(3, \text{blend}(6k, x, y))$ is a real number. Also, for $n \ge k$, $z_n = \text{blend}(6k, x, y)_{6n} \div 6 = y_{6n} \div 6 = \text{accel}(3, y)_n$. Therefore, $z \equiv \text{accel}(3, y)$ because $\text{bnddiff}(z, \text{accel}(3, y))$. By Corollary 11, $\text{accel}(3, y) \equiv y$, so $z \equiv y$. For $n < k$, $z_n = \text{blend}(6k, x, y)_{6n} \div 6 = x_{6n} \div 6 = \text{accel}(3, x)_n$, so $z = \text{accel}(3, x) \in \mathbb{S}_k$. □

Now we can prove the Main Lemma.

Proof. For any $x \in \mathbb{R}$, and any sequence g_n converging to x we can find a function c such that for all $k \in \mathbb{N}^+$, $|x - g_{c(k)}| \le 1/6k$. We define G by

$$G(k) = \text{accel}(3, \text{blend}(6k, x, g_{c(k)}))$$

Then, by Lemma 19, $G : k : \mathbb{N}^+ \to \{y : \mathbb{R} \mid y = \text{accel}(3, x) \in \mathbb{S}_k\}$, and for all $k \in \mathbb{N}^+$, $G(k) \equiv g_{c(k)}$.

Let $x' = \text{accel}(3, x)$. Then $x' \equiv x$ and by Lemma 17, for any $P \in \mathbb{R} \to \mathbb{B}$ there is a $k \in \mathbb{N}^+$ such that

$$P(G(k)) = P(x')$$

Then for $z = G(k)$ and $n = c(k)$ we have $P(z) = P(x')$, and $z \equiv g_n$.

□

That finishes the proof of the Main Lemma, and also the Main Theorem.

Corollary 20. *(Indecomposability of* $\mathbb{R}$*) If Set*(A) *and Set*(B) *and* $A \cap B = \emptyset$ *and* $A \cup B = \mathbb{R}$ *then* $(A = \mathbb{R}) \vee (A = \emptyset)$

Proof. If $A(0)$ then $(A = \mathbb{R}) \wedge (B = \emptyset)$ because for any $x \in \mathbb{R}$, $A(x) \vee B(x)$, and if $B(x)$ then by Theorem 1, $A \cap B \neq \emptyset$.

If $B(0)$ then $(B = \mathbb{R}) \wedge (A = \emptyset)$. □

Remark: Brouwer's proof of indecomposability used his theorem that all functions in $\mathbb{R} \to \mathbb{B}$ are constant. The proof of that theorem relies on both the continuity principle and the Fan theorem. We have proved the stronger connectedness property using only continuity. Loeb [3] had already shown that continuity alone is sufficient for this indecomposability result.

Mandelkern [5] shows that the contrapositive of Connected$((0,3))$ is not provable in INT. His example shows that even for open sets, U, V, we can not prove that if U and V are non-empty, disjoint, subsets of interval $(0,3)$ then there is a point in $(0,3)$ that is not in $U \cup V$.

6 Generalizations

If X is a subtype of $\mathbb{R}$, then for $A \in X \to \mathbb{P}$, we can define $\text{Set}_X(A)$ by $\forall x\!: X.\ \forall y\!: \{X|\ x \equiv y\}.\ P(x) \Rightarrow P(y)$ and $\text{Set}^+_X(A)$ by $(\text{Set}_X(A)\ \wedge\ \exists a\!:X.\ A(a))$ and $\text{Cover}^+_X(A,B)$ by $(\text{Set}^+_X(A)\ \wedge\ \text{Set}^+_X(A)\ \wedge\ \forall x\!:X.\ A(x) \vee B(x))$. Then we define Connected(X) by

$$\forall A, B : X \to \mathbb{P}.\ \text{Cover}^+_X(A,B) \Rightarrow (\exists x\!:X.\ A(x) \wedge B(x))$$

Theorem 1 is then the same as Connected$(\mathbb{R})$. Can we prove Connected(X) for some proper subtypes of $\mathbb{R}$? If X is an interval $(-\infty, b)$, $(-\infty, b]$, (a,b), $[a,b)$, $(a,b]$, $[a,b]$, (a,∞), or $[a,\infty)$ then the answer is yes.

To prove Connected$([a,b])$ we use the fact that

$$f_{a,b} = \lambda x.\ min(b, max(a,x))$$

is a retraction $\mathbb{R} \to [a,b]$. Then for $A, B\!: [a,b] \to \mathbb{P}$ we let $A' = \lambda x.\ A(f_{a,b}(x))$ and $B' = \lambda x.\ B(f_{a,b}(x))$. It is then easy to check that if $\text{Cover}^+_{[a,b]}(A,B)$ then $\text{Cover}^+_{\mathbb{R}}(A',B')$ so $\exists r\!:\mathbb{R}.\ A'(r)\ \wedge\ B'(r)$, and then $A(f_{a,b}(r))\ \wedge\ B(f_{a,b}(r))$.

Then to prove Connected(I) for any interval I we use

$$\text{Connected}([min(a,b), max(a,b)])$$

where $a \in I$ and $b \in I$ are the inhabitants of sets A and B.

Van Dalen [7] shows that the irrationals and the set $\mathbb{R} \setminus \{0\}$ are indecomposable using continuity, bar induction and Kripke's scheme, and Loeb [3, 4] shows that these indecomposability results follow from only continuity and a scheme, proposed by R.E. Vesley [8], that is weaker than Kripke's scheme. We can state Vesley's scheme in our terminology as follows:

If Set(X) define

$$\overline{X} = \{x : \mathbb{R} \mid \neg(x \in X)\}$$

Definition 21. (Vesley's Scheme) If Set(X) and $\overline{X}$ is dense in $\mathbb{R}$ then for any operation $\overline{f} \in \overline{X} \to \mathbb{N}$ there is an operation $f \in \mathbb{R} \to \mathbb{N}$ such that $f(x) = \overline{f}(x)$ for $x \in \overline{X}$, i.e. operation f is an *extension* of the operation $\overline{f}$.

Let VS be the formal statement of Vesley's scheme. We prove the following theorem that strengthens the results of van Dalen and Loeb:

Theorem 22. *If Set(X) and $\overline{X}$ is dense in $\mathbb{R}$ then VS $\Rightarrow$ Connected($\overline{X}$)*

Proof. We prove this like the main theorem 13. If non-empty sets A and B cover $\overline{X}$, and $\overline{X}$ is dense, then we can find $a_0 \in A$ and $b_0 \in B$ such that $a_0 < b_0 \ \vee \ b_0 < a_0$. Then without loss of generality we can assume $a_0 < b_0$. We have an operation d of type $d \in r : \overline{X} \to A(r) + B(r)$. We use this, as in the proof of Theorem 13, to construct sequences a_n and b_n in $\overline{X}$ that converge to a common limit $x \in \mathbb{R}$. In the proof of Theorem 13, at each step of the construction we considered the midpoint $m_n = (a_n + b_n)/2$ and chose a_{n+1}, b_{n+1} depending on whether $m_n \in A$ or $m_n \in B$. In this proof, the midpoint may not be in $\overline{X}$, so instead we use the density of $\overline{X}$ to find a point in the middle third of the interval (a_n, b_n).

Since $\overline{X}$ is dense, we can use Lemma 19 to find a real $x' \equiv x$ such that

$$(D) \qquad \forall k :: \mathbb{N}^+. \ \exists y : \overline{X}. \ y = x' \in \mathbb{S}_k$$

Since sequences converge to a unique limit, we have a_n converges to x' and b_n converges to x'.

Now we use Vesley's scheme to extend boolean operations $\lambda x.\ \mathrm{isl}(d(x))$ and $\lambda x.\ \mathrm{isr}(d(x))$ from $\overline{X} \to \mathbb{B}$ to $\mathbb{R} \to \mathbb{B}$ and call these extensions aa and bb. Since they are extensions, for $x \in \overline{X}$ we have $aa(x) = \mathrm{tt} \Rightarrow x \in A$ and $bb(x) = \mathrm{tt} \Rightarrow x \in B$.

Then Lemma 14 and its proof gives us n_1 and z_1 such that $z_1 \equiv a_{n_1}$ (so $z_1 \in A$) and $bb(z_1) = \mathrm{tt} \Leftrightarrow bb(\mathrm{accel}(3, x')) = \mathrm{tt}$. Similarly, we find n_2 and z_2 such that $z_2 \in B$ and $aa(z_1) = \mathrm{tt} \Leftrightarrow aa(\mathrm{accel}(3, x')) = \mathrm{tt}$.

Let $x'' = \text{accel}(3, x')$. We now claim that either $aa(x'') = \text{tt}$, in which case $z_2 \in A \cap B$, or $bb(x'') = \text{tt}$, in which case $z_1 \in A \cap B$.

The reason that either $aa(x'') = \text{tt}$ or $bb(x'') = \text{tt}$ is, again, the continuity principle, but we need the following slight strengthening of Axiom 15. It is also derivable from the stronger Nuprl continuity rule proved correct by Rahli.

Axiom 23. *(Weak Continuity+) For any $F, H \in \mathbb{S} \to \mathbb{B}$ and any $f \in \mathbb{S}$,*

$$\forall G\!:\!k\!:\!\mathbb{N}^+ \to \{g\!:\!\mathbb{S} \mid g = f \in \mathbb{S}_k\}.\ \exists k\!:\!\mathbb{N}^+.\ (F(G(k)) = F(f) \\ \wedge\ H(G(k)) = H(f))$$

Then using regularization we derive:

Lemma 24. *(Weak Continuity+ for $\mathbb{R}$) For $x \in \mathbb{R}$, and $P, Q \in \mathbb{R} \to \mathbb{B}$*

$$\forall G\!:\!k\!:\!\mathbb{N}^+ \to \{y\!:\!\mathbb{R} \mid y = x \in \mathbb{S}_k\}.\ \exists k\!:\!\mathbb{N}^+.\ P(G(k)) = P(x) \\ \wedge\ Q(G(k)) = Q(x)$$

For any $k \in \mathbb{N}^+$ we have, from (D), $\exists y : \overline{X}.\ y = x' \in \mathbb{S}_{6k}$. Then, since $x'' = \text{accel}(3, x')$, we have, for such y, $\text{accel}(3, y) = x'' \in \mathbb{S}_k$, and also, $\text{accel}(3, y) \in \overline{X}$ (because $y \in \overline{X}$, $\text{accel}(3, y) \equiv y$, and $\text{Set}(X)$). Thus, for every $k \in \mathbb{N}^+$ there is a $y_k \in \overline{X}$ such that $y_k = x'' \in \mathbb{S}_k$, so there is a $G \in \mathbb{N}^+ \to \{g\!:\!\mathbb{S} \mid g = x'' \in \mathbb{S}_k\}$ such that $\forall k.\ G(k) \in \overline{X}$.

Using Lemma 24, we get a $k \in \mathbb{N}^+$ such that $aa(G(k)) = aa(x'')$ and $bb(G(k)) = bb(x'')$. Since $G(k) \in \overline{X}$ we have $aa(G(k)) = \text{isl}(d(G(k))$ and $bb(G(k)) = \text{isr}(d(G(k)))$. So either $aa(G(k)) = \text{tt}$ or $bb(G(k)) = \text{tt}$, hence either $aa(x'') = \text{tt}$ or $bb(x'') = \text{tt}$. □

Remark It is not true that for any dense set X, real number x, and natural number k, there is a $y \in X$ that agrees exactly with x up to k (i.e. $y = x \in \mathbb{S}_k$). For example, the reader can check that the sequence x where $x_1 = -1$, $x_2 = 4$ and $x_n = n$ for $n > 2$ is regular. This real number x is equivalent to the rational $1/2$. Because the interval $\frac{x_1}{2} \pm \frac{1}{1}$ is $[-3/2, 1/2]$ and the interval $\frac{x_2}{4} \pm \frac{1}{2}$ is $[1/2, 3/2]$ any regular sequence y that agrees exactly with x up to $k = 2$ will be forced to satisfy $y \equiv 1/2$. Thus if X is the irrationals, we can not find a $y \in X$ that agrees exactly with x up to $k = 2$. We attempted simpler proofs of Theorem 22 but were thwarted by this fact. (Note that, for this example, $\text{accel}(3, x)$ is the sequence $\lambda n.\ n$ whose intervals are $1/2 \pm 1/n$, and now we can find irrational numbers that are in the first k of these intervals and agree with $\text{accel}(3, x)$ up to k).

Remark If Markov's principle holds, then $\mathbb{R}_{\neq 0} = \{x:\mathbb{R} \mid \neg(x \equiv 0)\}$ is not connected. But Theorem 22 shows that VS $\Rightarrow$ Connected($\mathbb{R}_{\neq 0}$). Thus VS contradicts MP. While Nuprl does not prove MP, we have shown that Nuprl is consistent with MP. Thus Nuprl can not prove VS. A future semantics for Nuprl may allow us to validate VS and hence reject MP, making Nuprl even more intuitionistic.

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