Noname manuscript No.
(will be inserted by the editor)

Implementing Euclid's Straightedge and Compass Constructions in Type Theory

Ariel Kellison · Mark Bickford · Robert Constable

Received: date / Accepted: date

Abstract Constructions are central to the methodology of geometry presented in the *Elements*. This theory therefore poses a unique challenge to those concerned with the practice of constructive mathematics: can the *Elements* be faithfully captured in a modern constructive framework? In this paper, we outline our implementation of Euclidean geometry based on straightedge and compass constructions in the intuitionistic type theory of the Nuprl proof assistant. A result of our intuitionistic treatment of Euclidean geometry is a proof of the second proposition from Book I of the *Elements* in its full generality; a result that differs from other formally constructive accounts of Euclidean geometry. Our formalization of the straightedge and compass makes use of a predicate for orientation, which enables a concise and intuitive expression of Euclid's constructions.

Keywords Constructive type theory, Nuprl, constructive geometry, foundations of geometry, constructive Euclidean geometry, intuitionistic geometry.

1 Introduction

The *Elements* of Euclid, now over two thousand years old, provided a paradigm for deductive proof until the axiomatic method became more fully developed in the 19th century. At this point a formalist tradition of mathematics was established and the notion of rigor was redefined. The axioms presented in the *Elements* were recast and reanalyzed in the formalist tradition first in Hilbert's 1899 *Foundations of Geometry* and then in Tarski's 1959 *What is Elementary Geometry?*. There is, however, a central feature of Euclid's methodology that

This material is based upon work supported by the National Science Foundation under Grant No.1650069.

Department of Computer Science, Cornell University
E-mail: {ak2485,msb367@cornell.edu}@cornell.edu, rc@cs.cornell.edu

is absent from the formal systems presented in both of these treatises: in the *Elements*, proving propositions and carrying out constructions are treated as identical activities [1].

From a logical point of view, constructions are utilized by Euclid in order to prove existential propositions where it is required to show that a certain geometric object exists. Thus, the original practice of Euclidean geometry can be interpreted as a specific instance of *intuitionistic mathematics*, for which *intuitionistic type theory* serves as a foundation. The *Elements* should therefore be expressible in this type theory. In this paper, we present an axiomatic treatment of Euclidean geometry, which we formalized in the intuitionistic type theory implemented by the Nuprl proof assistant.

In order to reason about the construction of geometric objects, Euclid takes construction postulates as primitive terms. These postulates can be interpreted as the basic admissible applications of the straightedge and collapsing compass (the collapsing compass *collapses* when it is lifted from the page). We formalized these tools in an intuitionistic way using an *apartness* predicate, written $\#$. Apartness is the positive analogue of inequality [2]. This predicate has been used in axiomatic theories of affine and projective geometry by Heyting [3], van Dalen [4,5], Mandelkern [6] and von Plato [7,8]. Apartness is applied to Euclidean geometry for the first time in the present work. The introduction of the apartness predicate to Euclidean geometry provides an additional tool for the intuitionistic geometer. We refer to this tool as the *magnifying glass*.

Our utilization of the apartness predicate and the magnifying glass differentiate our interpretation of admissible Euclidean construction procedures from those proposed by Beeson [9,10]. Our intuitionistic approach enables a proof of Proposition 2 from Book I of the *Elements* in its full generality. In Proposition 2, the *non-collapsing compass* is constructed from the primitive collapsing compass and straightedge. A fully general proof is not provable in Beeson's interpretation of constructive Euclidean geometry.

Proof assistants like Nuprl [11] that implement constructive type theory (of which intuitionistic type theory is a variant) operate under the *proofs-as-programs* paradigm [12], in which there is an isomorphism between proofs and programs. Thus, for any of Euclid's propositions that require the construction of a geometric object, Nuprl will generate an executable program that corresponds to the proof. Our Nuprl programs for the propositions from the first book of the *Elements* are fairly straightforward to follow by hand using a straightedge and compass or in software tools for geometry visualizations. The intuitive simplicity of a program corresponding to a geometric construction proof is not guaranteed for *any* formalization of the straightedge and compass. We found it necessary to include a predicate for *orientation* in our formalization of these tools. Utilization of this predicate enabled concise and intuitive proofs.

2 Related Work

The question "What is constructive geometry?" has multiple answers. This is a point which Vesley [13] elucidates well. This section will introduce works on constructive geometry that we believe are most similar to our own. We will additionally introduce other works that present formalizations of geometry in proof assistants.

Constructive approaches to geometry. The first intuitionistic axiomatization of geometry was given by Heyting [14] in his 1925 thesis on the foundations of projective geometry. Heyting extended his work on projective geometry by providing an axiom system for affine geometry in order to effect a projective extension [3]; van Dalen continued this work [5, 4]. Mandelkern [6] similarly explores projective geometry in the spirit of Bishop [15]. Intuitionistic accounts of affine geometry expressed in type theory are given by von Plato [7]. Included in von Plato's work on affine geometry [8] is also a predicate for a point *apart left* from a line, as well as the notion of *directed* lines. This is similar to the *leftness* predicate utilized in this work. Our axioms for *leftness* were influenced by our reading of Knuth's text on computational geometry, *Axioms and Hulls* [16].

While all of the literature mentioned so far make use of the intuitionistic property of apartness, none focus on Euclidean geometry. In this direction is the intuitionistic counterpart to Tarski's account of Euclidean geometry given by Lombard & Vesley [17]. Works that specifically focus on the straightedge and compass constructions found in the *Elements* are given by Mäenpää and von Plato [1] and Beeson [9,18].

Mäenpää and von Plato do not give an axiomatic account of Euclidean geometry but instead emphasize the correspondence between Euclid's construction procedures and intuitionistic type theory. Their work gives support to our reading of the *Elements*, and influences Section 3 of this work, where we introduce those aspects of intuitionistic type theory that enable a faithful expression of Euclid's methodology within a formal framework.

Beeson's constructive axioms for Tarski's geometry [19] motivated the first Nuprl implementation of the *Elements* [20]. Beeon's work, and therefore the Nuprl implementation, do not utilize apartness; the *stability of equality* is taken as an axiom instead. The original Nuprl formalization did not aim to build constructors based on the straightedge and collapsing compass as we do in this work. A main constructor in the theory is instead a version of the Axiom of Segment Extension (Axiom 4 in *Metamathematische Methoden in der Geometrie*), which we discuss in Section 5.2.

Formalizations of geometry in proof assistants. Formalizations of *Metamathematische Methoden in der Geometrie* (Schwabäuser, Szmielew, & Tarski, [21]) have been implemented in the Coq proof assistant [22,23] and the OTTER theorem prover [24]. A formalization of Hilbert's *Foundations of Geometry* has been implemented in the proof assistant Isabelle [25].

It has already been mentioned that parts of Beeson's work have been formalized before in Nuprl.

Mandelkern's constructive account of projective geometry [6] has been formalized in Agda [26]. Finally, perhaps most similar to the work we present here is Kahn's [27] implementation of von Plato's account of intuitionistic geometry in type theory.

3 Intuitionistic Foundations for Geometry

A logical foundation for the practice of geometry presented in the *Elements* must be able to express constructions explicitly [1]. It must also be able to capture the correspondence between proving a proposition and carrying out a construction. In this section, we introduce those aspects of intuitionistic type theory that enable a faithful expression of Euclid's methodology. This section also supplies terminology and necessary background for the axiomatization presented in Section 4.

3.1 Intuitionistic Type Theory

In intuitionistic type theory, the explicit expression of constructions emerges as a result of the correspondence between propositions and the types of evidence for those propositions [28]. This correspondence is referred to by the dictum *propositions-as-types* [29]. Specifically, for any proposition A, there is a corresponding a that is an object of the type of proofs of A, written $a : A$. Thus, conjunction $A \mathbin{\&} B$ corresponds to the Cartesian product $A \times B$ and a proof of the proposition $A \mathbin{\&} B$ consists of a proof of A and a proof of B, i.e. the pair (a, b). Disjunction $A \vee B$ corresponds to the disjoint sum $A + B$, and a proof of the proposition $A \vee B$ consists of either an $a : A$ or a $b : B$ along with evidence that either a or b belong to one of the disjuncts. Furthermore, implication $A \supset B$ corresponds to a function space $A \rightarrow B$, and a proof of the proposition $A \supset B$ consists of a function that with input a generates some b.

Universal and existential propositions correspond to *dependent types*. The existential proposition $\exists x : A.\ B(x)$ corresponds to the dependent product $x : A \times B(x)$, the proof of which consists of the pair (a,b) where $a : A$ and $b : B(a)$. The universal proposition $\forall x : A.\ B(x)$ corresponds to the dependent function space $x : A \rightarrow B(x)$, the proof of which consists of a function that with input $a : A$ generates some $b : B(a)$. Such a proof can be expressed as a lambda term in a functional programming language. In the case of both $A \supset B$ and $\forall x : A.\ B(x)$, the lambda term is $\lambda a.b$.

The type theoretic interpretation of the logical constructors makes evident the differences in the acceptable instances of proof in intuitionistic and classical logic. For example, proving the proposition $\exists x.\ P(x)$ using the impossibility of $\forall x. \neg P(x)$ is not intuitionistically acceptable.

The computational interpretation of the logic provided by intuitionistic type theory allows us to handle constructions explicitly. It also allows us to capture the correspondence between proving a proposition and carrying out a construction by extending the notion of *propositions-as-types* to the *proofs-as-programs* paradigm previously mentioned. Specifically, the proof of a proposition constructs an inhabitant of its type. The method of construction is a program; we will refer to *programs* and the *computational content* of a proof interchangeably in this work.

Proof assistants like Nuprl that implement intuitionistic type theory will generate (extract) a program from a proof. These proof *extracts* are the computational content of a proof. We supply the Nuprl extracts from our proofs of propositions from the *Elements* in Section 5.

3.2 The Computational Content of Euclid's Propositions

Not all propositions from the first book of the *Elements* require the construction of a geometric object for their proof. For example, the first proposition, *to construct and equilateral triangle on a finite straight line*, does require an explicit construction, while the sixth proposition, *if in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another*, does not. The common practice is to differentiate between these two forms of propositions by referring to those that require an explicit construction as *problems* and those that do not as *theorems*([30], p. 703). While the proof of a theorem may include auxiliary constructions, the main goal is not to construct an object with specific characteristics, but to demonstrate that certain relations hold for a given object.

In order to effect constructions in the *Elements*, three *construction postulates* corresponding to the basic admissible applications of the straightedge and collapsing compass are taken as primitive terms. Euclid [31] postulates the following:

1. *To draw a straight line from any point to any point.*
2. *To produce a finite straight line continuously in a straight line.*
3. *To describe a circle with and centre and distance.*

Logically, problems from the *Elements* are of the form $\forall x : A \, \exists y : B. \, C(x, y)$ while theorems are of the form $\forall x : A. \, C(x)$. The proof of a problem is a function that, from a given geometric object a of type A $(a : A)$, generates a pair consisting of a geometric object $b : B(a)$ and a proof c that the relation $C(a, b)$ holds. The proof of a theorem is a function that, from a given geometric object a of type A, generates a proof c that the relation $C(a)$ holds. In order to prove Euclid's theorems and problems using intuitionistic type theory, we must first formalize Euclid's construction postulates.

4 Intuitionistic Construction Postulates and Axioms

4.1 The Predicates

We take only one type of primitive geometric object: points. On points we take as primitive the binary *apartness* relation and the ternary *leftness* relation. *Leftness* establishes the notion of *orientation* on our Euclidean Plane.

Definition 1 A *Euclidean Plane* structure has a primitive type Point together with the following relations for any $a, b, c, d \in$ Point.

Congruence, written $ab \cong cd$, says that segments ab and cd have the same length.
Betweenness, written a_b_c, says that the point b lies between a and c. This relation is not *strict*, so b could be equivalent to either a or c.
Apartness is a binary relation, signified by #, on points. If $a\#b$ we say that a is *separated* from b.
Leftness is a ternary relation on points, written a left of bc, and says that the point a is to the $left$ of the line bc (by bc we mean the *directed* line from b to c).

In an oriented Euclidean plane, lines are directed:

Definition 2 For any $a, b, and\ c \in$ Point we say that a is to the *right of cb* in case a is to the left of bc:

$$a \text{ right of } cb \Leftrightarrow_{\text{def}} a \text{ left of } bc$$

In addition to the notion of orientation, *leftness* implies strict separation between a point and a line:

Definition 3 For any $a, b, and\ c \in$ Point a is strictly *separated from the line* bc if either a left of bc or a right of bc:

$$a\#bc \Leftrightarrow_{\text{def}} (a \text{ left of } bc \vee a \text{ right of } bc).$$

Definition 4 The primitive type Point has its own equality, written $a = b \in$ Point, or just $a = b$. But none of our axioms use this relation. Instead, our axioms imply that the negation of the apartness relation on points is an equivalence relation on points, and that the other primitive relations respect this equivalence:

$$a \equiv b \Leftrightarrow_{\text{def}} \neg(a\#b).$$

For any a, b, and $c \in$ Point we say that a, b, and c are *collinear* if a is not separated from the line bc:

$$\text{collinear}(a, b, c) \Leftrightarrow \neg a\#bc. \tag{Ax 1}$$

4.2 The Constructive Content of the Predicates

We exploit the implicit constructive content in the leftness and apartness predicates in our Nuprl implementation using Nuprl's *squash type*. The squash, $\downarrow P$, of a proposition P is the type where the witness for P is replaced by a fixed constant.

A proposition P is *squash stable* when $\downarrow P \to P$. In other words, just knowing that there is a witness for P but not given such a witness, we can compute P.

Apartness and leftness are squash stable propositions [1]:

$$\forall a, b\colon\text{Point.}\ \downarrow(a\#b) \to a\#b \tag{Ax 2}$$

$$\forall a, b, c\colon\text{Point.}\ \downarrow(a\#bc) \to a\#bc \tag{Ax 3}$$

So, for example, knowing only $\downarrow a\#b$ we can compute a witness to the separation $a\#b$ using only the points a and b.

The congruence and betweenness relations are *stable* propositions, they have no constructive content. A stable proposition P is one for which $\neg\neg P \to P$ holds. Classically, this is true for all propositions; in intuitionistic logic it only holds for some. For congruence and betweenness we take the axioms below.

$$\forall a, b, c, d\colon\text{Point.}\ \neg\neg(ab \cong cd) \to ab \cong cd \tag{Ax 4}$$

$$\forall a, b, c\colon\text{Point.}\ \neg\neg(a\_b\_c) \to a\_b\_c \tag{Ax 5}$$

We also fix length comparison for segments as a stable proposition:

Definition 5 For any a, b, c, and $d \in$ Point we say that segment cd is *at least as long* as segment ab when b lies between a and some point x and ax is equal in length to cd:

$$cd \geq ab \Leftrightarrow_{\text{def}} \neg\neg(\exists x.\ a\_b\_x \wedge ax \cong cd).$$

[1] Apartness and leftness need not be squashed. However, squashing these propositions simplifies proofs because we don't have to "carry around" the corresponding proof witnesses for squashed propositions. More specifically, for any a and b, if $a\#b$ is a squash stable proposition, then with "hidden" evidence for $a\#b$ we can prove $a\#b$:

$$a : \text{Point}, b : \{c : \text{Point} \mid a\#c\} \vdash a\#b.$$

Here we exploit Nuprl's *set type*[[32],p.12] in order to hide the evidence for $a\#b$ in b. If $a\#b$ were not a squash stable proposition, the proof of $a\#b$ would require explicit evidence:

$$a : \text{Point}, b : \text{Point}, c : a\#b \vdash a\#b.$$

4.3 The Postulates

In this section, we introduce our formalization of the primitive construction postulates from the *Elements*. Intuitively, the straightedge and collapsing compass can be *applied* to given inputs (points) just as a function is applied. Thus, all of our postulates have a functional reading which makes clear their computational content; we will refer to our postulates in this form as *constructors*. In addition to three basic admissible applications of the straightedge and collapsing compass, our postulates include a tool unique to the intuitionistic geometer: the *magnifying glass*.

The Straightedge-Straightedge Postulate Given $a, b, x, y \in$ Point, with $a\#b$, x left of ab, and
y right of ab, the Straightedge-Straightedge Postulate corresponds to the straightedge constructions of the segments ab and xy, and their unique point of intersection, z, as in Figure 1. The point z is constructed between x and y, collinear with ab:

$$\forall a, b, x, y\!:\!\text{Point. } (x \text{ left of } ab \wedge y \text{ right of } ab) \rightarrow \\ \exists z\!:\!\text{Point. } x\_z\_y \wedge \text{collinear}(z, a, b). \quad \text{(Ax 6)}$$

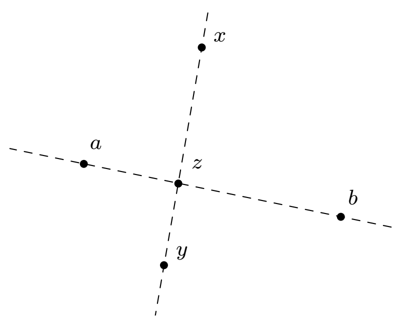


Fig. 1 The Straightedge-Straightedge (SS) Postulate determines the point of intersection for lines ab and xy given that x left of ab and y right of ab.

The Straightedge-Straightedge Constructor We let $SS(a, b, x, y)$ be the point z constructed by the *Straightedge-Straightedge* axiom.

The Straightedge-Compass Postulate Given $a, b, c, d \in$ Point with $a\#b$ and c_b_d, the Straightedge-Compass Postulate extends the segment ab constructed by the straightedge using the compass construction of the circle with center c and radius cd ($Circle(c, d)$) as in Fig. 2. If b is separated from d, then b is strictly inside $Circle(c, d)$. In this case, two separated intersection points result, call them u and v. Both u and v lie on line ab

and circle $Circle(c, d)$; b lies between u and v. Along the line ab, either a_b_u or a_b_v. Our formalization fixes u such that a_b_u:

$$\forall a,b,c,d\!:\!\text{Point}.\ (a\#b\ \wedge\ c\_b\_d) \rightarrow \\ \exists u,v\!:\!\text{Point}.\ cu \cong cd\ \wedge\ cv \cong cd\ \wedge\ a\_b\_u\ \wedge\ v\_b\_u \\ \wedge\ \text{collinear}(a,b,v)\ \wedge\ (b\#d \rightarrow u\#v). \tag{Ax 7}$$

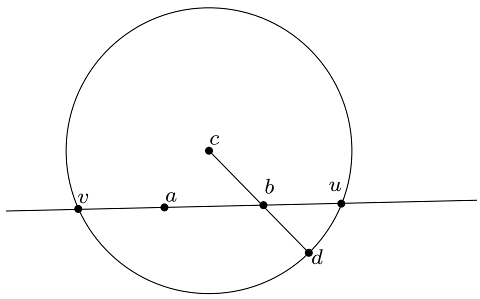


Fig. 2 A general configuration for the Straightedge-Compass (SC) Postulate. The non-strict betweenness relation c_b_d allows cases, including $c \equiv b$.

The Straightedge-Compass Constructor We define functions SCO(a, b, c, d) and SCS(a, b, c, d) to construct two points u and v respectively such that the relations s_b_u and b_a_v hold.

The Compass-Compass Postulate Given $a, b, c, d \in$ Point with $a\#c$, the Compass-Compass Postulate constructs $Circle(a, b)$ and $Circle(c, d)$. If these circles *overlap* then there are possibly equivalent points u and v lying on the circumferences of both circles. If the circles *strictly overlap* then u and v lie on opposite sides of ac. We fix u left of ac (see Figure 3).

$$\forall a,b,c,d\!:\!\text{Point}.\ (a\#c\ \wedge\ \text{Overlap}(a,b,c,d)) \rightarrow \\ \exists u,v\!:\!\text{Point}.\ au \cong ab\ \wedge\ cu \cong cd\ \wedge\ av \cong ab\ \wedge\ cv \cong cd \\ \wedge\ \text{StrictOverlap}(a,b,c,d) \rightarrow (u \text{ left of } ac\ \wedge\ v \text{ right of } ac). \tag{Ax 8}$$

Definition 6 For any $a, b, c,$ *and* $d \in$ Point, the two circles $Circle(a, b)$ and $Circle(c, d)$ *overlap* if there exist a p *and* $q \in$ Point such that p is on $Circle(a, b)$ and in $Circle(c, d)$ and q is on $Circle(c, d)$ and in $Circle(a, b)$.

$$\text{Overlap}(a,b,c,d) \Leftrightarrow \downarrow\exists p,q.\ ab \cong ap \wedge cd \geq cp \wedge cd \cong cq \wedge ab \geq aq$$

Definition 7 For any $a, b, c,$ *and* $d \in$ Point, the two circles $Circle(a, b)$ and $Circle(c, d)$ *strictly overlap* if there exist a p *and* q $\in$ Point such that p is on $Circle(a, b)$ and *strictly* in $Circle(c, d)$ and q is on $Circle(c, d)$ and *strictly* in $Circle(a, b)$.

$$\text{StrictOverlap}(a, b, c, d) \Leftrightarrow \downarrow\exists p, q.\ ab \cong ap \wedge cd > cp \wedge cd \cong cq \wedge ab > aq.$$

We need to know that points p and q exist but p and q are not needed to construct the points u and v, so Definition 6 and Definition 7 can be squashed.

The Compass-Compass Postulate corresponds to the Circle-Circle Continuity Principle, which defines the conditions for points of intersection between two circles [31]. In addition to constructing the intersection points of overlapping circles, our Compass-Compass Postulate contains the *leftness* predicate, which *orients* the points of intersection with respect to the line formed by the centers of the circles.

The Compass-Compass Constructor The functions $\text{CCL}(a, b, c, d)$ and $\text{CCR}(a, b, c, d)$ construct from circles $Circle(a, b)$ and $Circle(c, d)$ a point to the left of ac and a point to the right of ac, respectively.

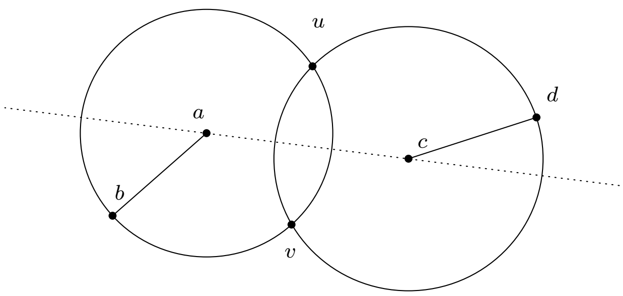


Fig. 3 The *Compass-Compass* constructor for the points u and v such that u left of ac and v right of ac: $u = \text{CCL}(a, b, c, d)$ *and* $v = \text{CCR}(a, b, c, d)$.

The Magnifying Glass Introduced as an axiom by Heyting and often referred to as the "co-transitivity of apartness," the magnifying glass postulate states that given two separated points a and b, any point c is separated from either a or b.

$$\forall a, b, c\!:\!\text{Point}.\ (a \# b) \rightarrow (c \# a \vee c \# b). \qquad \text{(Ax 9)}$$

We refer to this axiom as the "magnifying glass" because we can think of magnifying the region around the point a. Since b is separated from a, we can increase magnification on a until b is out of our region of sight. When this happens, if c is also outside the region then $c\#a$. Otherwise $c\#b$.

The Magnifying Glass Constructor $M(a, b, c)$, is the function from Ax 9 that, by *magnification*, decides whether $c\#a$ or $c\#b$.

Remark on the magnifying glass We could make equivalence coincide with equality by using a quotient type Point// $\equiv$. In this case the magnifying glass postulate would give, for $a\#b$, a function that decides on input c whether $c\#a$ or $c\#b$. This function would have to respect the equivalence relation. A corollary of Brouwer's uniform continuity theorem is that all such functions are constant on the real numbers. This implies that we would not be able to prove that the plane constructed from the real numbers satisfies the magnifying glass postulate. This illustrates Bishop's claim that forming the quotient is "either pointless or incorrect"[[15], p.65]. If equality were decidable, the quotient would be pointless. For the real plane, it is incorrect. This clarifies Beeson's claim that co-transitivity is not continuous. Specifically, the Magnifying Glass Postulate can't respect equivalence and therefore is not a well defined function on the quotient Point// $\equiv$.

Non-triviality The existence of two separated points is guaranteed by the non-triviality axiom:

$$\exists a, b\!:\!\text{Point}.\ a\#b. \qquad \text{(Ax 10)}$$

The Non-trivial Constructor We let O and X be the primitive separated points in our theory that are guaranteed by axiom Ax 10.

The Remaining Axioms The axioms Ax 11 - Ax 26, listed in Figure 4, are universally quantified and contain no disjunctions or existential quantifiers. These axioms therefore do not construct anything but, instead, state that some relation holds when other relations hold.

Origins of the Axioms Several of our axioms are taken from other axiomatizations of geometry. Co-transitivity of apartness (Ax 9) was introduced to geometry by Heyting. From Tarski's work, we have the upper dimension axiom (restricting our geometry to a plane) (Ax 21), non-triviality (Ax 10), the five segment axiom (Ax 20), and the transitivity of congruence (Ax 19).

We also take an axiom of "cyclic symmetry" (Ax 22) about the leftness predicate from Knuth's *Axioms and Hulls*, in which Knuth develops an axiomatic approach to convex hull algorithms. Although we have not yet explored this idea further, we have proved (in Nuprl) that Knuth's axioms follow from our own. Adoption of "cyclic symmetry" reduced the number of axioms in our system.

$$\neg(a\#a) \tag{Ax 11}$$
$$(cd \geq ab \ \wedge \ a\#b) \rightarrow c\#d \tag{Ax 12}$$
$$a \equiv b \rightarrow a\_b\_c \tag{Ax 13}$$
$$a\_b\_c \rightarrow c\_b\_a \tag{Ax 14}$$
$$(a\_b\_d \ \wedge \ b\_c\_d) \rightarrow a\_b\_c \tag{Ax 15}$$
$$(a \equiv b \ \wedge \ c\_a\_d) \rightarrow c\_b\_d \tag{Ax 16}$$
$$aa \cong bb \tag{Ax 17}$$
$$a \equiv b \rightarrow ac \cong cb \tag{Ax 18}$$
$$(ab \cong cd \ \wedge \ ab \cong pq) \rightarrow cd \cong pq \tag{Ax 19}$$
$$(FS(a,b,c,d,A,B,C,D) \ \wedge \ a\#b) \rightarrow cd \cong CD \tag{Ax 20}$$
$$(x\#y \wedge xa \cong ya \wedge xb \cong yb \wedge xc \cong yc) \rightarrow \text{collinear}(a,b,c) \tag{Ax 21}$$
$$a \text{ left of } bc \rightarrow b \text{ left of } ca \tag{Ax 22}$$
$$a \text{ left of } bc \rightarrow b\#c \tag{Ax 23}$$
$$(x \text{ left of } ab \ \wedge \ y \text{ left of } ab \ \wedge \ x\_z\_y) \rightarrow z\#ab \tag{Ax 24}$$
$$(a\#bc \ \wedge \ y\#b \ \wedge \ \text{collinear}(y,a,b)) \rightarrow y\#bc \tag{Ax 25}$$
$$(cx \cong cy \ \wedge \ ca \cong cb \ \wedge \ c\_b\_y \ \wedge \ c\_x\_a) \rightarrow a \equiv x \tag{Ax 26}$$

Free variables a,b,c,d, A,B,C,D, x,y,z,p,q : Point are universally quantified. The *five segment configuration*, $FS(a,b,c,d,A,B,C,D)$, used in the *five segment axiom* (Ax 20), is defined to be

$$a\_b\_c \ \wedge \ A\_B\_C \ \wedge \ ab \cong AB \ \wedge \ bc \cong BC \ \wedge \ ad \cong AD \ \wedge \ bd \cong BD$$

Fig. 4 The remaining axioms for constructive plane geometry.

Remark on Leftness We utilized the *leftness* relation in our straightedge and compass postulates in order to address two issues we encountered during our formalization of the *Elements*. Specifically, Euclid implicitly constructs points of intersection (between lines, between circles, and between circles and lines) and also takes for granted the fact that one can distinguish between (potentially) multiple points of intersection. For example, Proposition 1 of Book I is *to construct an equilateral triangle on a given finite straight line*. Firstly, the given construction assumes the intersection points of two circles. Secondly, the given construction yields two equilateral triangles, only one of which need be chosen as evidence for the proof. Such choices need to be accounted for in a formal proof. By formalizing the straightedge and collapsing compass using *leftness*, we were able to ameliorate both of these issues in a succinct way.

5 Extracting Constructive Content from Euclid's Propositions

From the *proofs-as-programs* principle it is possible for us to represent our proofs as Nuprl extracts [2]. From any proof about a Euclidean Plane (Definition

[2] Our complete Nuprl proofs can be found online `http://www.nuprl.org/LibrarySnapshots/Published/Version2/Mathematics/euclidean!plane!geometry/index.html`

1) satisfying our axioms, Nuprl will automatically *extract* the constructive content, which corresponds to the constructors defined in Section 4.3. The Nuprl extract will be a function of the inputs (the Euclidean Plane e which we treat as implicit input, the universally quantified points, and witnesses for any constraints) and will produce the witnesses for existentially quantified points using the constructors. Our Nuprl proofs are intuitive, and are fairly straightforward to implement by hand using a straightedge and compass or in a geometry visualization software tool. In this way, we retain the intuitive nature of proofs from the *Elements*.

5.1 Propositions and Programs

Propositions 4,5,6, 7, and 8 are *theorems* and therefore do not develop any new constructions, but instead verify that some properties of the Euclidean plane hold. In this section, we detail our proofs of Propositions 1 and 2, which are *problems* and thus contain constructive content. For Propositions 3,9, and 10, we only list the Nuprl extracts.

We begin with Euclid's first proposition, which constructs an equilateral triangle:

To construct an equilateral triangle on a given finite straight line.

Our formal statement of Proposition 1 supposes that $a\#b$ is the given (non-degenerate) "finite straight line." We include an extra assertion requiring the construction of a non-degenerate equilateral triangle, with an apex that lies *to the left* of ab.

Proposition 1

$$\forall a\!:\!Point.\ \forall b\!:\!\{Point \mid b\#a\}.\ \exists c\!:\!\{Point \mid \ Cong3(a,b,c) \wedge c \ left\ of\ ab\}$$

where

$$Cong3(a,b,c) = ab \cong bc \ \wedge \ bc \cong ca \ \wedge \ ca \cong ab.$$

We easily prove Proposition 1 as Euclid does by using the Compass-Compass (CC) Postulate with circles $Circle(a,b)$ and $Circle(b,a)$. This constructs two equilateral triangles, and we take the one where c left of ab.

Our Nuprl extract reflects the simplicity of the construction:

$$\lambda a.\lambda b.\ \mathrm{CCL}(a,b,b,a).$$

So we can define

$$\triangle(a,b) = CCL(a,b,b,a)$$

as the program for Euclid's proposition 1, where CCL is the *Compass-Compass left* constructor from Section 4.3. See Figure 5.

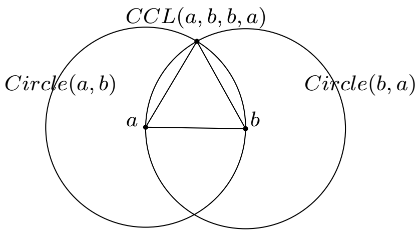


Fig. 5 Proposition 1.

Euclid's Proposition 2 proves that the non-collapsing compass can be constructed using the collapsing compass (our primitive compass) and straightedge:

> *To place a straight line equal to a given straight line with one end at a given point.*

We can prove Proposition 2 from our axioms using a slightly more complex proof than Euclid's. We use Euclid's original construction as a lemma with the given point a and the given straight line bc. In the lemma, we require that $a\#b$:

Lemma 1

$$\forall a\!:\!Point.\ \forall b\!:\!\{Point|b\#a\}.\ \forall c\!:\!Point.\ \exists d\!:\!\{Point|ad \cong bc\}.$$

Proof Euclid's proof of the lemma is to first construct an equilateral triangle abx using Proposition 1, which requires $a\#b$. Then the point $u = \mathrm{SCO}(x, b, b, c)$ from the straightedge-compass (SC) axiom satisfies x_b_u and $bu \cong BC$. Then $d = \mathrm{SCS}(a, x, x, u)$ from SC once more to construct satisfies $XD \cong XU$ and either x_a_d or x_d_a because d and a are on the same side of x. We use Ax 26 to rule out x_d_a, so we have x_a_d. Since x_b_u and $xd \cong xu$ and $xa \cong xb$, it follows from a lemma derivable from the five segment axiom (Ax 20) that $ad \cong bc$ as required.

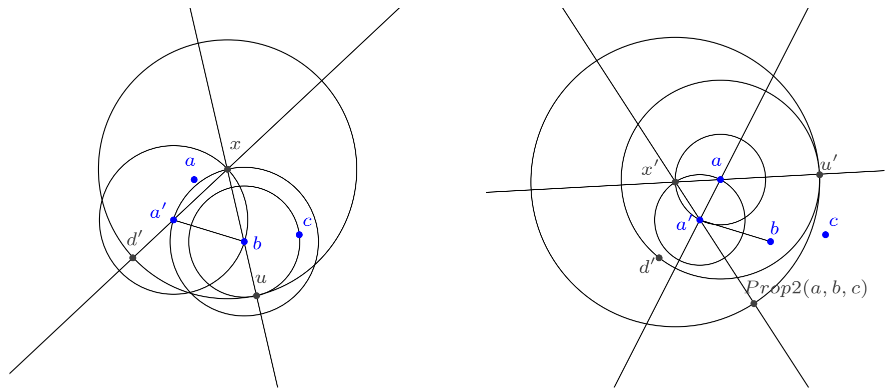
The program extracted from this lemma is

$$\begin{aligned}\mathrm{lmma2}(a, b, c) = &\mathrm{let}\ x = \triangle(a, b)\ \mathrm{in}\\ &\quad\mathrm{let}\ u = \mathrm{SCO}(x, b, b, c)\ \mathrm{in}\\ &\quad\quad\mathrm{SCS}(a, x, x, u)\end{aligned}$$

We utilize Lemma 1 for the unrestricted Proposition 2, where $a\#b$ is not required.

Proposition 2

$$\forall a, b, c\!:\!Point.\ \exists d\!:\!\{Point|ad \cong bc\}$$



(a) The application of lmma2(a',b,c) to construct the point d'. Here $X = \triangle(a', b)$ and $u = \mathrm{SCO}(x, b, b, c)$.

(b) The application of lmma2(a',a,d') to construct the point $Prop2(a, b, c)$. Here $x' = \triangle(a', a)$ and $u' = \mathrm{SCO}(x, a, a, d')$.

Fig. 6 The general form of Proposition 2 in the case when $a\#b$, $a\#a'$, and $b\#a'$.

Proof We must construct d without assuming $a\#b$. Because by Ax 10 we have separated points $o\#x$, then by Ax 9 we have either $a\#o$ or $a\#x$ so we have a point a' separated from a. Then, using Ax 9 again we have $b\#a \vee b\#a'$. If $b\#a$ we use the lemma. If $b\#a'$ then we use Lemma 1 to construct D' with $a'd' \cong bc$, and then, since $a'\#a$, we use Lemma 1 as lmma2(a',a,d') to construct d with $ad \cong a'd'$. Then $ad \cong a'd' \cong bc$ as required (using Ax 19).

Figure 6 provides a visual for Proposition 2 in the case when $a\#b$, $a\#a'$, and $b\#a'$. The program extracted from Proposition 2 is

$$\begin{aligned}\mathrm{Prop2}(a,b,c) = &\text{if } M(o,x,a)\\ &\quad \text{then if } M(a,o,b) \text{ then } \mathrm{lmma2}(a,b,c)\\ &\qquad \text{else } \mathrm{lmma2}(a,o,\mathrm{lmma2}(o,b,c))\\ &\quad \text{else if } M(a,x,b) \text{ then } \mathrm{lmma2}(a,b,c)\\ &\qquad \text{else } \mathrm{lmma2}(a,x,\mathrm{lmma2}(x,b,c)).\end{aligned}$$

Remaining Propositions Propositions 3, 9, 10, 11, and 12 also contain constructive content. Below, we list the informal statements of these Propositions (taken from the *Elements* [31]) along with their formal Nuprl statements and their Nuprl extracts.

Extension Operator (see Section 5.2):

$$\text{extend } qa \text{ by } bc = \mathrm{SCO}(q, a, a, \mathrm{Prop2}(a, b, c)).$$

Proposition 3: To cut off from the greater of two given unequal straight lines a straight line equal to the less.

$$\begin{aligned}\text{Prop3}(a, b, c_1, c_2) = \text{let } X = \text{extend } ba \text{ by } c_1c_2 \text{ in}\\ \text{extend } xa \text{ by } c_1c_2\end{aligned}$$

Proposition 10: To bisect a given finite straight line.

$$\text{Mid}(a, b) = \text{let } p = \text{CC}(a, b, b, a) \text{ in SS}(b, a, \text{snd}(p), \text{fst}(p))$$

Proposition 9: To bisect a given rectilinear angle.

$$\text{AngleBisect}(a, c, b) = \text{Mid}(\text{extend } ca \text{ by } cb, \text{extend } cb \text{ by } ca)$$

Proposition 11: To draw a straight line at right angles to a given straight line from a given point on it.

$$\begin{aligned}\text{ErectPerp}(a, b, c) = &\text{if } M(a, b, c)\\ &\text{then let } x = \text{SCO}(c, a, a, c) \text{ in}\\ &\quad \text{Mid}(\text{SCO}(b, a, c, x), \text{SCS}(b, a, c, x))\\ &\text{else let } x = \text{SCO}(c, b, b, c) \text{ in}\\ &\quad \text{Mid}(\text{SCO}(a, b, c, x), \text{SCS}(a, b, c, x))\end{aligned}$$

Proposition 12 : To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

$$\begin{aligned}\text{DropPerp}(a, b, c) = &\text{if } M(a, b, c)\\ &\text{then let } x = \text{SCO}(a, c, c, a) \text{ in } \triangle(a, x)\\ &\text{else let } x = \text{SCO}(c, b, b, c) \text{ in } \triangle(b, x)\end{aligned}$$

(Where $\triangle$ is the program for Proposition 1.)

It can be seen from the extracts that in our theory, the sequence of the propositions is not the same as in the *Elements*. For example, we were able to prove Proposition 10 before Proposition 9.

Propositions 4, 5, 6, 7, and 8 do not have any constructive content (they are *theorems*), but their proofs follow from our axioms and the proofs of *problems* previously presented. Figure 7 lists the informal (taken from the *Elements* [31]) and formal statements of these propositions. The full Nuprl proofs for any of our theorems can be found at http://www.nuprl.org/LibrarySnapshots/Published/Version2/Mathematics/euclidean!plane!geometry/index.html.

Proposition 4. If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.
Proposition 5. In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.
Proposition 6. If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.
Proposition 7. Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.
Proposition 8. If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

Fig. 7 Propositions 4, 5, 6, 7, and 8 do not have any constructive content.

5.2 Tarski's Axiom of Segment Extension and Proposition 2

Tarski's Axiom of Segment Extension [33] was used by Sernaker and Constable, following Beeson, in previous implementations of the *Elements* in Nuprl. We restate Tarski's Axiom here using the notation for betweenness and congruence already introduced, lowercase Roman letters still denote points:

$$\exists x((q\_a\_x) \ \wedge \ ax \cong bc).$$

Thus, given any line segment bc, a line segment (ax) can be constructed congruent to it, as in Figure 8.

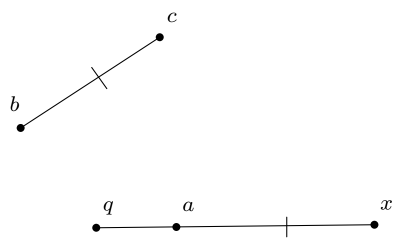


Fig. 8 Tarski's Axiom of Segment Extension.

If $q \equiv a$ then the Axiom of Segment Extension corresponds to Proposition 2 as given in Section 5.1. If $a\#q$, Proposition 2 can be used to copy segment bc to point q and then the Straightedge-Compass Postulate can be used to construct the required point x. But, unlike our proof of Proposition 2, we can not prove the general form of the Segment Extension Axiom because that would require us to decide $(a\#q) \vee \neg(a\#q)$, which is not an intuitionistically valid choice. So we must be content with an extend operation for $a\#q$:

Theorem 1

$$\forall q, b, c\colon Point.\ \forall a\colon \{Point|a\#q\}.\ \exists x\colon \{Point|q\_a\_x \wedge ax \cong bc\}$$

.

The extract is

$$\text{extend}(A, B, C, D) = \text{SCO}(A, B, B, \text{Prop2}(B, C, D)).$$

6 Model of the Axioms

We verified the consistency of our axioms in Nuprl using the constructive reals, where real numbers are constructive Cauchy Sequences [34] and Point $\in \mathbb{R}^2$. Building this model did not require use of the law of the excluded middle or Markov's principle [3]. Our axioms do not assume decidable equality and points in the constructive reals do not have decidable equality. Indeed, our axioms do not mention equality at all.

On $\mathbb{R}^2$ we can define a distance function

$$d(x, y) = \sqrt{(x_0 - y_0)^2 + (x_1 - y_1)^2}.$$

The interpretation of $x\#y$ is $d(x, y) > 0$, and the interpretation of $ab \cong cd$ is $d(a, b) = d(c, d)$. Because in our model the real numbers are constructive Cauchy sequences, the relation $=$ is an equivalence relation on the Cauchy sequences. This equivalence is a stable proposition (i.e. $\neg\neg(x = y) \to (x = y)$, holds). The defintion of $<$ on the real numbers has the form $\exists n : \mathbb{N}.\ P(n)$ for a decidable P, so it is *squash* stable. Markov's Principle would imply that $<$ is stable, but we do not need MP, because we assume that apartness and leftness are *squash stable*, not *stable*.

The interpretation of a left of bc is $\det(a, b, c) > 0$ where $\det(a, b, c)$ is the determinant

$$\det(a, b, c) = \begin{vmatrix} a_0 & a_1 & 1 \\ b_0 & b_1 & 1 \\ c_0 & c_1 & 1 \end{vmatrix}.$$

Then, because $\det(a, c, b) = -\det(a, b, c)$,

$$a\#bc \Leftrightarrow |\det(a, b, c)| > 0.$$

We define $B(a, b, c)$ to be $\exists t \in (0, 1).\ b = t * a + (1 - t) * c$. The somewhat non-obvious interpretation of a_b_c is then

$$a\_b\_c \Leftrightarrow \neg(a\#b \wedge b\#c \wedge \neg(B(a, b, c) \wedge a\#c))$$

[3] Markov's Principle (MP) says that propositions of the form $\exists n : \mathbb{N}.\ P(n)$ are stable when $P(n)$ is decidable. Without MP such propositions are squash stable but not stable.

which, being a negation, is a stable proposition.

Using these definitions for the primitive propositions, we verified (in Nuprl) all of the axioms for constructive Euclidean plane geometry given in Figure 4 [4].

The *algebraic real numbers*, $\mathbb{A}$, have decidable equality. Thus, when points have type $\mathbb{A}^2$ one gets a model of our axioms. In this model it would be provable that $(a\#b) \Leftrightarrow \neg(a = b)$, so it would be possible to have axioms that do not use apartness. But the standard Euclidean plane based on $\mathbb{R}^2$ would not, constructively, satisfy such axioms.

7 Concluding Remarks

Euclid's methodology for geometry lends itself to formalization in intuitionistic type theory in a manner that modern influential axiomatic theories of plane Euclidean geometry, such as those presented by Hilbert and Tarski, do not. Specifically, both the *Elements* of Euclid and intuitionistic type theory have an embedded correspondence between proofs and constructions.

A main goal of our axiomatization was to formalize the straightedge and collapsing compass so that their constructive content intuitively expresses constructions in the *Elements*. We discovered that an orientation predicate, which we call *leftness*, enabled such a formalization.

A previous Nuprl implementation of the *Elements* followed Beeson's constructive axioms [19] for Tarski's geometry [20]. This formalization makes use of the Axiom of Segment Extension described in Section 5.2. In *Metamathematische Methoden in der Geometrie* Tarksi et. al. go to great lengths in order to prove circle-circle continuity, which corresponds to our Compass-Compass Postulate, as a theorem. We were not able to find a constructive proof of this theorem. Instead, we took *leftness* as a predicate and Compass-Compass as a postulate, which greatly simplified our system.

We took the constructive predicate of *apartness* in our axiomatization, which had not yet been utilized in a formalization of Euclidean geometry. The co-transitivity of apartness, which we refer to as the Magnifying Glass (Ax 9), is an additional tool available to the intuitionistic geometer that enabled our proof of Proposition 2 from Book I of the *Elements* in its full generality. Beeson [9] has objected to the use of the apartness predicate and the co-transitivity axiom in formalizing the *Elements*, and therefore proves only an instance of Proposition 2. In our theory, equivalence and equality on points do not coincide, so the constructive real numbers serve as a model that satisfies co-transitivity.

Finally, the theory we present in this paper could be formalized in other proof assistants that implement constructive logic. Only the details of the proof automation would be different. Specifically, although we used some of

[4] Our model can be found online: `http://www.nuprl.org/LibrarySnapshots/Published/Version2/Mathematics/reals!model!euclidean!geometry/index.html`

Nuprl's types, such as the squash type and set types in our formalization in order to obtain more intuitive extracts, these are non-essential [5].

References

1. P. Mäenpää and J. von Plato, "The logic of Euclidean construction procedures," *Acta Philos. Fenn*, vol. 49, pp. 275–293, 1990.
2. D. van Dalen, "Heyting A.. Axiomatic method and intuitionism. Essays on the foundations of mathematics dedicated to A. A. Fraenkel on his seventieth anniversary, edited by Bar-Hillel Y., Poznanski E. I. J., Rabin M. O., and Robinson A. for the Hebrew University of Jerusalem, Magnes Press, Jerusalem 1961, and North-Holland Publishing Company, Amsterdam 1962, pp. 237247." *The Journal of Symbolic Logic*, vol. 36, no. 03, pp. 522–523, 9 1971. [Online]. Available: https://www.cambridge.org/core/product/identifier/S0022481200082475/type/journal_article
3. A. Heyting, "Axioms for Intuitionistic Plane Affine Geometry," *Studies in Logic and the Foundations of Mathematics*, vol. 27, pp. 160–173, 1959. [Online]. Available: http://linkinghub.elsevier.com/retrieve/pii/S0049237X09700266
4. D. van Dalen, "Outside as a primitive notion in constructive projective geometry," *Geometriae Dedicata*, vol. 60, no. 1, pp. 107–111, 3 1996. [Online]. Available: http://link.springer.com/10.1007/BF00150870
5. D. V. Dalen, "Extension Problems in Intuitionistic Plane Projective Geometry." [Online]. Available: https://www.illc.uva.nl/Research/Publications/Dissertations/HDS-15-Dirk-van-Dalen.text.pdf
6. M. Mandelkern, "A constructive real projective plane," *Journal of Geometry*, vol. 107, no. 1, pp. 19–60, 4 2016. [Online]. Available: https://doi.org/10.1007/s00022-015-0272-4
7. J. von Plato, "The axioms of constructive geometry," *Annals of Pure and Applied Logic*, vol. 76, no. 2, pp. 169–200, 12 1995. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0168007295000052
8. ——, "A constructive theory of ordered affine geometry," *Indagationes Mathematicae*, vol. 9, no. 4, pp. 549–562, 12 1998. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0019357798800347
9. M. Beeson, "Constructive Geometry," in *Proceedings of the 10th Asian Logic Conference*, 2009, pp. 19–84. [Online]. Available: http://www.worldscientific.com/doi/abs/10.1142/9789814293020_0002
10. ——, "Brouwer and Euclid," *Indagationes Mathematicae*, vol. 29, no. 1, pp. 483–533, 2 2018. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0019357717300447
11. R. L. Constable, S. F. Allen, H. M. Bromley, W. R. Cleaveland, J. F. Cremer, R. W. Harper, D. J. Howe, T. B. Knoblock, N. P. Mendler, P. Panangaden, J. T. Sasaki, and S. F. Smith, *Implementing Mathematics with the Nuprl Proof Development System*, 1985. [Online]. Available: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.55.4216
12. R. L. Constable, "Programs as proofs: a synopsis," *Information Processing Letters*, vol. 16, no. 3, pp. 105–112, 4 1983. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0020019083900601
13. R. Vesley, "Constructivity in Geometry," *History and Philosophy of Logic*, vol. 20, no. 3-4, pp. 291–294, 10 1999. [Online]. Available: http://www.tandfonline.com/doi/abs/10.1080/01445349950044206
14. A. Heyting, "Zur intuitionistischen Axiomatik der projektiven Geometrie," *Mathematische Annalen*, vol. 98, no. 1, pp. 491–538, 3 1928. [Online]. Available: http://link.springer.com/10.1007/BF01451605
15. E. Bishop, *Foundations of Constructive Analysis.* New York: McGraw-Hill, 1967.
16. D. E. Knuth, *Axioms and hulls.* Springer-Verlag, 1992. [Online]. Available: https://books.google.com/books/about/Axioms_and_hulls.html?id=vghRAAAAMAAJ

[5] See Section 4.2

17. M. Lombard and R. Vesley, "A common axiom set for classical and intuitionistic plane geometry," *Annals of Pure and Applied Logic*, vol. 95, no. 1-3, pp. 229–255, 11 1998. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0168007298000177
18. M. Beeson, "Logic of Ruler and Compass Constructions." Springer, Berlin, Heidelberg, 2012, pp. 46–55. [Online]. Available: http://link.springer.com/10.1007/978-3-642-30870-3_6
19. ——, "A constructive version of Tarski's geometry," *Annals of Pure and Applied Logic*, vol. 166, no. 11, pp. 1199–1273, 11 2015. [Online]. Available: http://linkinghub.elsevier.com/retrieve/pii/S0168007215000718
20. Sarah Sernaker and Robert L. Constable, "Formal Exploration of Geometry," 2016. [Online]. Available: http://www.nuprl.org/MathLibrary/geometry/
21. W. Schwabhäuser, W. Szmielew, and A. Tarski, *Metamathematische Methoden in der Geometrie*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1983. [Online]. Available: http://link.springer.com/10.1007/978-3-642-69418-9
22. P. Boutry, C. Gries, J. Narboux, and P. Schreck, "Parallel Postulates and Continuity Axioms: A Mechanized Study in Intuitionistic Logic Using Coq," *Journal of Automated Reasoning*, pp. 1–68, 9 2017. [Online]. Available: http://link.springer.com/10.1007/s10817-017-9422-8
23. J. Narboux, "Mechanical Theorem Proving in Tarskis Geometry," in *Automated Deduction in Geometry*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006, pp. 139–156. [Online]. Available: http://link.springer.com/10.1007/978-3-540-77356-6_9
24. M. Beeson and L. Wos, "OTTER Proofs in Tarskian Geometry," in *Automated Reasoning*, S. Demri, D. Kapur, and C. Weidenbach, Eds. Cham: Springer International Publishing, 2014, pp. 495–510.
25. L. I. Meikle and J. D. Fleuriot, "Formalizing Hilberts Grundlagen in Isabelle/Isar," 2003, pp. 319–334. [Online]. Available: http://link.springer.com/10.1007/10930755_21
26. G. Calderón, "Formalizing constructive projective geometry in Agda," in *LSFA 2017: the 12th Workshop on Logical and Semantic Frameworks, with Applications*, Brasília, 2017, pp. 150–165. [Online]. Available: http://lsfa2017.cic.unb.br/LSFA2017.pdf
27. G. Kahn, "Constructive Geometry according to Jan von Plato," vol. V5.10, 1995.
28. R. L. Constable, "The Semantics of Evidence," Cornell University, Ithaca, NY, Tech. Rep., 1985.
29. P. Wadler, "Propositions As Types," *Commun. ACM*, vol. 58, no. 12, pp. 75–84, 11 2015. [Online]. Available: http://doi.acm.org/10.1145/2699407
30. J. Avigad, E. Dean, and J. Mumma, "A Formal System for Euclid' s Elements," *The Review of Symbolic Logic*, vol. 2, no. 4, 2009. [Online]. Available: http://repository.cmu.edu/philosophy
31. T. Heath, *The thirteen books of Euclid's Elements*. New York: Dover, 1956.
32. S. Allen, M. Bickford, R. Constable, R. Eaton, C. Kreitz, L. Lorigo, and E. Moran, "Innovations in computational type theory using Nuprl," *Journal of Applied Logic*, vol. 4, no. 4, pp. 428–469, 12 2006. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S1570868305000704
33. A. Tarski and S. Givant, "Tarski's System of Geometry," *Bulletin of Symbolic Logic*, vol. 5, no. 02, pp. 175–214, 6 1999. [Online]. Available: https://www.cambridge.org/core/product/identifier/S1079898600007010/type/journal_article
34. M. Bickford, "Constructive Analysis and Experimental Mathematics using the Nuprl Proof Assistant," 2016. [Online]. Available: http://www.nuprl.org/documents/Bickford/reals.pdf